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Adjustment Grid Methods

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ABSTRACT

Within the market data approach to real estate appraisal, two basic
types of analysis generally are used: (1) regression analysis; and,
(2) adjustment grid methods. The focus of this paper is on the adjust-
ment grid methods. Three such methods are identified in the appraisal
literature, but their analytical foundations are not clearly presented.
The primary objective of this paper is to clarify the analytical foun-
dations of each method. In addition, various ways to estimate the ad-
justment factors needed to apply the grid methods and alternative
weighting schemes for reconciliation of indicated values are presented.
Also, the possible advantage of grid based over purely regression based
predictions is identified.

The Analytical Foundations of Adjustment Grid Methods

INTRODUCTION

Three adjustment grid methods are usually identified in the appraisal literature. Unfortunately, there is no coherent presentation of the analytical foundations of these methods that allows for a properly informed choice of method. The purposes of this paper are to clarify the analytical foundations of each method, to demystify the two of the three methods that are little understood and little used, to offer suggestions for the derivation of adjustment factors and weighting schemes, and to compare predictions based on grid methods to their chief competitor among market approaches, pure regression predictions.

Each grid method is based on a functional relationship between selling price and property attributes commonly called a hedonic price function. The rationale for the use of hedonic price functions to estimate the price of housing has been developed by Rosen.[1] Here we develop the adjustment grid methods. First, some of the misconceptions in the appraisal literature about such methods are summarized. Second, the specific form of each method's hedonic price function is presented. Third, various ways to estimate the adjustment factors needed to apply the grid methods are indicated. Fourth, alternative weighting schemes for reconciliation of indicated values are presented. Finally, the possible advantage of grid based predictions over purely regression based predictions is identified.

THE LITERATURE ON THE GRID METHODS

There is some confusion in the appraisal literature about the analytical foundations of the adjustment grid methods. A key aspect

of this confusion is the failure of many to recognize that each of the methods presumes a specific functional relationship of selling price with property attributes. The three adjustment grid methods discussed are: (1) Additive Dollar Adjustments Method (ADAM), (2) Additive Percentage Adjustments Method (APAM), and (3) Multiplicative Percentage Adjustments Method (MPAM).[2]

In the terms of this paper, the APAM and the ADAM may share the same hedonic price function as is commonly believed to be the case.[3] However, the APAM may differ from the ADAM in some respects, including its underlying hedonic price function. The literature is correct in indicating that the adjustments in the APAM are independent whereas they are interdependent for the MPAM.[4] In more specific terms the cross partial derivatives of the hedonic price function underlying the APAM are zero, whereas they are nonzero for the MPAM.

The reasons given for choosing one method over others illustrate some of the confusion in the literature. For example, it is said that the ADAM is generally superior and preferred because the market thinks and acts in terms of dollars[5] or because small adjustments are difficult to measure by percentages.[6] The reasons given for not using the MPAM vary from the vague (e.g., it is no longer valid[7] or it is a very poor alternative[8]) to the specific (e.g., there is a tendency to overadjust for comparative deficiencies and to underadjust for superior features[9]). None of these reasons are valid.

There are other suggestions in the literature that are proper but are never accompanied by precise instructions as to how they can be

utilized. It is suggested, for example, that weights used in the recon-
ciliation of different indicated values reflect the number, magnitude,
and reliability of adjustments.[10] However, there has been no attempt to
prescribe precisely how these weights might be constructed to properly
reflect the relevant factors and to show what impact such a weighting
scheme has on the estimation of market value.

While it is widely understood that the adjustment factors should
reflect the marginal contribution of the property attributes, the lit-
erature is misleading or silent with regard to the estimation of the
adjustment factors used in the grid methods. With respect to the
ADAM, the literature offers precise instruction (i.e., use matched pairs)
that leads to statistically unreliable estimates. There is virtually no
guidance regarding the estimation of the adjustment factors implicit in
the other two methods.

The problems found in the literature are removed in the remainder
of this paper. As indicated, the APAM may differ from the ADAM in
some respects. The MPAM is divisible into two rather distinct
methods, each with its own underlying hedonic price function. The
validity and possible bias of each method is shown to depend on the
accuracy of the specification of its underlying hedonic price function.

ANALYTICAL FOUNDATIONS

The application of the grid methods is based on functional relation-
ships of selling price with property attributes that may vary across or
within methods. Use of the Additive Dollar Adjustments Method (ADAM)
presumes a linear relationship. Use of the Additive Percentage Adjustments

Method (APAM) may or may not presume that the relationships are linear. However like the ADAM, it definitely presumes that there is no interaction among the effects of the property attributes on value. In contrast, use of the Multiplicative Percentage Adjustments Method (MPAM) presumes that the partial relationships between selling price and the attributes may be non-linear and that there are interaction effects. Specifically, the MPAM may be based on an exponential function or on a function that is of the form economists call Cobb- ouglas.

Additive Dollar Adjustments Method

Use of the Additive Dollar Adjustments Method (ADAM) is based on the assumption that selling price is a linear function of property attributes. Assuming for the sake of simplicity that only two attributes vary substantially, such a function is as follows:

(1) $$SP_i = f_0 + f_1 X_{1i} + f_2 X_{2i} + \alpha_i$$

where SP_i = the actual selling price of the ith property,

X_{ji} = the jth attribute of the ith property (For example, X_{1i} might be the lot area of the ith property),

f_j = the adjustment factor for the jth attribute, and

α_i = the error term associated with the ith property.

The actual selling price is explained by equation (1) which includes a random error term on the right side. The predicted selling price is derived by omitting the error term, as follows:

(2) $$\widehat{SP}_i = f_0 + f_1 X_{1i} + f_2 X_{2i}$$

where $\widehat{SP}_i$ = the predicted selling price of the ith property.

Thus for comparable 1,

(3) $$\widehat{SP}_1 = f_0 + f_1 X_{11} + f_2 X_{21}.$$

For the subject property,

(4) $$\widehat{SP}_s = f_0 + f_1 X_{1s} + f_2 X_{2s}$$

where $\widehat{SP}_s$ = the predicted selling price of the subject property, and

X_{js} = the jth attribute of the subject property.

Subtracting equation (3) from equation (4) yields:

(5) $$\widehat{SP}_s - \widehat{SP}_1 = f_1(X_{1s}-X_{11}) + f_2(X_{2s}-X_{21}).$$

Equation (5) indicates that the difference in selling prices between the subject and the comparable equals an adjustment factor multiplied by the difference in one attribute plus another adjustment factor multiplied by the difference in another attribute, etc. This is consistent with the Additive Dollar Adjustments Method (ADAM) and f_1 and f_2 are the adjustment factors one would wish to use in the ADAM.

Utilizing (5), the ADAM can be characterized as computing the indicated value as follows:

(6) $$\widehat{SP}_s^1 = SP_1 + (\widehat{SP}_s - \widehat{SP}_1)$$

where $\widehat{SP}_s^1$ = the predicted selling price or indicated value of the subject property based on a comparison with comparable 1, and

SP_1 = the actual selling price of comparable 1.

In order to discover the meaning of the indicated value computed in this fashion, it is necessary to break down the comparable's selling

price into its constituent parts and substitute into equation (6). From equations (1) and (2), it is clear that:

(7) $$SP_i = \widehat{SP}_i + \alpha_i.$$

Thus for comparable 1,

(8) $$SP_1 = \widehat{SP}_1 + \alpha_1.$$

Substituting equation (8) into equation (6) yields:

(9) $$\widehat{SP}_s^1 = \widehat{SP}_1 + \alpha_1 + (\widehat{SP}_s - \widehat{SP}_1).$$

Simplifying,

(10) $$\widehat{SP}_s^1 = \widehat{SP}_s + \alpha_1.$$

The indicated selling price for the subject based on comparable 1 and computed via the ADAM is the sum of the predicted selling price for the subject from equation (4) and the portion of comparable 1's actual selling price that is unexplained by the model.

Additive Percentage Adjustments Method

The Additive Percentage Adjustments Method (APAM) may be rationalized by a number of functional forms. In general, the percentage difference in prices between a subject and a comparable is found by adding functions of the two properties' attributes. Next, the estimate of the percentage difference is multiplied by the price of the comparable to find the price difference. Finally, the price difference is added to the price of the comparable to find the indicated value of the subject by the adjustments of the comparable. Symbolically, this may be represented as follows:

(11) $\dfrac{\widehat{SP}_s - \widehat{SP}_1}{SP_1} = r_{11}(X_{1s}, X_{11}) + r_{21}(X_{2s}, X_{21}),$

where r_{11} and r_{21} denote general expressions for alternative functional forms that are consistent with the APAM. The indicated selling price for the subject based on comparable 1 is then found as follows:

(12) $\widehat{SP}_s^1 = SP_1 + SP_1 \left[\dfrac{\widehat{SP}_s - \widehat{SP}_1}{SP_1} \right].$

The functional forms denoted by r_{11} and r_{21} include such things as a constant multiplied by differences in attributes, by differences in reciprocals of attributes, by differences in logs of attributes or by differences in powers of attributes. The only thing really excluded by practice is interaction terms across attributes. The hedonic functions underlying these functions require selling price to be a linear, reciprocal, log or power function of the attributes, respectively. Of course, it is possible for various combinations of these functions to be used.

Linear Function - If a linear function is presumed, using equation (2) a percentage difference between the selling price of the subject property and that of a comparable property (say comparable 1) can be expressed as follows:[11]

(13) $\dfrac{\widehat{SP}_s - \widehat{SP}_1}{SP_1} = \dfrac{f_1 X_{1s}}{SP_1} \left[\dfrac{(X_{1s} - X_{11})}{X_{1s}} \right] + \dfrac{f_2 X_{2s}}{SP_1} \left[\dfrac{(X_{2s} - X_{21})}{X_{2s}} \right].$

Let the adjustment factors for equation (13) be g_{11} and g_{21} as follows:

(14) $g_{11} = \dfrac{f_1 X_{1s}}{SP_1}, \quad g_{21} = \dfrac{f_2 X_{2s}}{SP_1}$

Equation (2) underlies the Additive Dollar Adjustments Method (ADAM),
whereas equation (13), which is derived from (2), provides a specific
functional form for equation (11), the heart of the Additive Percentage
Adjustments Method (APAM). This indicates that the APAM does not differ
in any significant way from the ADAM if a linear hedonic price function
underlies the ADAM. However, it can be seen from equation (14) that
there is no adjustment factor that is valid across all comparables for
the APAM. The adjustment factor must incorporate the selling price of
the comparable in order to derive an adjustment expressed as a percentage.

Non-Linear Functions - If a non-linear hedonic price function under-
lies the APAM, there are significant differences between the APAM and the
ADAM. For example, a log hedonic price function is expressed as follows:

$$(15) \qquad \widehat{SP}_i = b_0 + b_1 \ln X_{1i} + b_2 \ln X_{2i},$$

where b_0, b_1, b_2 are parameters.

Using equation (15), a percentage difference between the subject property
and a comparable property (say comparable 1) can be expressed as follows:[12]

$$(16) \qquad \frac{\widehat{SP}_s - \widehat{SP}_1}{SP_1} = \frac{b_1}{SP_1} \ln \left[\frac{X_{1s}}{X_{11}} \right] + \frac{b_2}{SP_1} \ln \left[\frac{X_{2s}}{X_{21}} \right].$$

Again, the adjustment factor ($g_{ji} = b_j/SP_i$) must incorporate the
selling price of the comparable in order to derive an adjustment expressed
as a percentage. A similar procedure could be followed to incorporate the
reciprocals of attributes or the powers of attributes.

Finally, it can be shown that a Cobb-Douglas hedonic price function
is consistent with the Additive Percentage Adjustments Method only in the
limit. Consider a function as follows:

(17) $\qquad SP_i = h_0 \; X_{1i}^{h_1} \; X_{2i}^{h_2} \; e^{u_i}$

where h_j = the adjustment factor for the jth attribute,

$\qquad$ $e \approx 2.718 \ldots$ (base of natural logarithms), and

$\qquad$ u_i is the error term associated with the ith property.

Dropping the error term so as to get predicted selling price and taking

the natural logarithm of equation (17) yields:

(18) $\qquad \ln \widehat{SP}_i = \ln h_0 + h_1 \ln X_{1i} + h_2 \ln X_{2i}.$

Differentiating equation (18) yields:

(19) $\qquad \dfrac{d\widehat{SP}_i}{\widehat{SP}_i} = h_1 \dfrac{dX_{1i}}{X_{1i}} + h_2 \dfrac{dX_{2i}}{X_{2i}}$

Equation (19) indicates that the percentage difference in selling prices

(e.g., the difference between the subject and a comparable) equals a

factor multiplied by the percentage difference in one attribute plus

another factor multiplied by the percentage difference in another

attribute. Thus, the Cobb-Douglas function is consistent with the

Additive Percentage Adjustments Method in the limit, meaning for infin-

itesimally small differences between the subject and the comparable.

It will be shown in the next section that the Cobb-Douglas form of

hedonic function is completely consistent with the Multiplicative

Percentage Adjustments Method.

Multiplicative Percentage Adjustments Method

$\qquad$ What is known about this method is that the ratio of the selling

price of the subject to that of the comparable is equal to the product

of functions of the subject and comparable's attributes. There is no

direction in the literature as to what specific functions these are. In general, the indicated value of the subject is found as follows:

$$(20) \qquad \widehat{SP}_s^1 = SP_1 (\widehat{SP}_s / \widehat{SP}_1).$$

There are two distinct functional forms that provide rationalizations for the Multiplicative Percentage Adjustments Method. These are the Cobb-Douglas and the exponential forms.

Cobb-Douglas Function - For the Cobb-Douglas function [See equation (17).], dropping the error term and dividing the predicted selling price of the subject by the predicted selling price of comparable 1 yields:

$$(21) \qquad \widehat{SP}_s / \widehat{SP}_1 = (X_{1s}/X_{11})^{h_1} (X_{2s}/X_{21})^{h_2}.$$

Substituting equation (21) into equation (20) yields:

$$(22) \qquad \widehat{SP}_s^1 = SP_1 (X_{1s}/X_{11})^{h_1} (X_{2s}/X_{21})^{h_2}.$$

Equation (22) indicates that the predicted selling price of the subject property is equal to the selling price of the comparable property, multiplied by the product of functions of the subject's and comparable's attributes.

Exponential Function - Again assuming that only two attributes vary substantially, an exponential function can be written as:

$$(23) \qquad SP_i = h_0 e^{h_1 X_{1i}} e^{h_2 X_{2i}} e^{v_i}$$

where h_j = the adjustment factor for the jth attribute, and

v_i is the error term associated with the ith property.

The predicted selling price is derived by omitting the error term, as follows:

$$(24) \qquad \widehat{SP}_i = h_0 e^{h_1 X_{1i}} e^{h_2 X_{2i}}.$$

Thus for comparable 1,

$$(25) \qquad \widehat{SP}_1 = h_0 e^{h_1 X_{11}} e^{h_2 X_{21}}.$$

For the subject property,

$$(26) \qquad \widehat{SP}_s = h_0 e^{h_1 X_{1s}} e^{h_2 X_{2s}}.$$

Dividing equation (26) by equation (25) yields:

$$(27) \qquad \frac{\widehat{SP}_s}{\widehat{SP}_1} = e^{h_1 (X_{1s} - X_{11})} e^{h_2 (X_{2s} - X_{21})}.$$

Substituting equation (27) into equation (20) yields:

$$(28) \qquad \widehat{SP}_s^1 = SP_1 e^{h_1 (X_{1s} - X_{11})} e^{h_2 (X_{2s} - X_{21})}.$$

Equation (28) indicates that the predicted selling price of the subject property is equal to the selling price of the comparable property, multiplied by a term related to the difference in one attribute, then multiplied by a term related to the difference in another attribute. These multiplicative terms represent percentage adjustments. This is consistent with the Multiplicative Percentage Adjustments Method and h_1 and h_2 are the adjustment factors one would wish to use for this Method.

In order to discover the meaning of the indicated value computed in this fashion, it is necessary to break down the comparable's selling price into its constituent parts and substitute into equation (28). From equations (23) and (24), it can be shown that:

(29) $SP_i = \widehat{SP}_i e^{v_i}.$

Thus for comparable 1,

(30) $SP_1 = \widehat{SP}_1 e^{v_1}.$

Substituting equation (30) into equation (20) yields:

(31) $\widehat{SP}_s^1 = \widehat{SP}_1 e^{v_1}(\widehat{SP}_s/\widehat{SP}_1)$

Simplifying,

(32) $\widehat{SP}_s^1 = \widehat{SP}_s e^{v_1}$ (Similarly for the Cobb-Douglas form,
        $\widehat{SP}_s^1 = \widehat{SP}_s \, e^{u_1}$).

The indicated selling price for the subject property based on comparable 1 is the product of the predicted selling price for the subject from equation (26) and the proportion of comparable 1's actual selling price that is unexplained by the model.

ESTIMATION OF ADJUSTMENT FACTORS

There are at least three ways to estimate the adjustment factors needed for the grid methods. These three are matched pairs, regression analysis, and the use of cost data.

Matched Pairs

Some direction is given in real estate appraisal texts on the use of matched pairs.[13] It is generally suggested that pairs of comparable sales be found which differ only with respect to the attribute for which an adjustment factor is sought. The factor is then a function of the selling prices and the magnitudes of that one attribute. For the Additive Dollar Adjustments Method (ADAM) the function can be expressed as the difference in selling prices divided by the difference in the one attribute which differs between the two properties, as follows:

$$(33) \qquad f_j = \frac{SP_1 - SP_2}{X_{j1} - X_{j2}}$$

where f_j = the adjustment factor for the jth attribute,

SP_1 and SP_2 are the selling prices of the matched pair, and

X_{j1} and X_{j2} are the magnitudes of the jth attribute for the matched pair.

There is no adjustment factor that is valid across all comparables for the Additive Percentage Adjustments Method (APAM). The adjustment factor must incorporate the selling price of the comparable in order to derive an adjustment expressed as a percentage [See equations (13) and (14).]. Assuming a linear underlying hedonic price function, the adjustment factor for the APAM (for the ith comparable) is derived from the adjustment factor for the ADAM by multiplying f_j by the magnitude of the jth attribute of the subject property and dividing by the selling price of the ith comparable as follows:

$$(34) \qquad g_{ji} = \frac{f_j X_{js}}{SP_i}$$

If f_j is derived on the basis of a matched pair as in equation (33), then g_{ji} also is based on a matched pair.

For the Multiplicative Percentage Adjustments Method (MPAM) the function for the adjustment factor is expressed as follows, if the underlying hedonic price function is Cobb-Douglas:

$$(35) \qquad h_j = \frac{\ln SP_1 - \ln SP_2}{\ln X_{j1} - \ln X_{j2}}$$

where h_j = the adjustment factor for the jth attribute.

Alternatively, if the underlying hedonic price function is exponential the function for the adjustment factor is expressed as follows:

$$(36) \qquad h_j = \frac{\ln SP_1 - \ln SP_2}{X_{j1} - X_{j2}} \ .$$

Regression Analysis

One problem frequently cited for the use of matched pairs is the difficulty in finding a pair of sales which differ significantly in only one attribute. Another problem with deriving adjustment factors by using paired sales is that there are no degrees of freedom. That is, the matched pair approach determines a straight line relationship between selling price (or natural logarithm of selling price) and each property attribute (or natural logarithm of the attribute). At minimum, two observations on each attribute are needed for this determination. One more would provide one degree of freedom. Two more (or four in all) would provide two degrees of freedom, etc. Having some degrees of freedom allows one to place some statistical confidence in adjustment factors. Thus, each adjustment factor can be derived by the estimated slope of a simple regression between selling price (or natural logarithm of selling price) and the attribute (or

natural logarithm of the attribute) which varies among a number (more than two) of otherwise similar properties. More than two are needed in order to have confidence in the estimates. When sufficient sales differing only with respect to a single attribute are not available, as is generally the case, multiple regression can be used to estimate adjustment factors. A relatively large number of sales of comparable properties is needed for this technique, but none need to be identical to others in any attribute.

The regression equation which must be estimated for the Additive Dollar Adjustments Method may be expressed as follows:

(37) $SP = f_0 + f_1 X_1 + f_2 X_2 + \ldots + f_m X_m.$

The adjustment factor for the Additive Percentage Adjustments Method (APAM) may be derived by transforming the adjustment factors from equation (37) if the underlying hedonic price function is assumed to be linear. This transformation is the same as that shown for equation (34).

However, if the underlying hedonic price function is non-linear the adjustment factors must be derived on the basis of a non-linear regression equation. For example, if a log function underlies the APAM, the regression equation may be expressed as follows:

(38) $SP = b_0 + b_1 \ln X_1 + b_2 \ln X_2 + \ldots + b_m \ln X_m$

Similarly, a non-linear regression equation could be developed on the basis of reciprocals or powers.

For the Multiplicative Percentage Adjustments Method (MPAM) the regression equation which must be estimated if the underlying hedonic price function is Cobb-Douglas is expressed as follows:

$$(39) \quad \ln SP = \ln h_0 + h_1 \ln X_1 + h_2 \ln X_2 + \ldots + h_m \ln X_m.$$

Alternatively, the regression equation which must be estimated for the MPAM if the underlying function is exponential is expressed as follows:

$$(40) \quad \ln SP = \ln h_0 + h_1 X_1 + h_2 X_2 + \ldots + h_m X_m.$$

Cost Data

Cost data provides a means to estimate adjustment factors if the housing market is in equilibrium. Considerable guidance is given in real estate appraisal texts on the sources of cost data.[14] These sources include local contractors and national cost estimating services.

For the Additive Dollar Adjustments Method (ADAM) the adjustment factor is derived from cost data as follows:

$$(41) \quad f_j' = \frac{C_1 X_{j1} - C_2 X_{j2}}{X_{j1} - X_{j2}}$$

where X_{j1} and X_{j2} include the range of the jth attribute found in the comparables and the subject,

C_1 = cost per unit of the jth attribute for an observation with X_{j1} units of that attribute, and

C_2 = cost per unit of the jth attribute for an observation with X_{j2} units of that attribute.

Again, the adjustment factor for the Additive Percentage Adjustments Method (APAM) may be derived by transforming the adjustment factors for the ADAM if the underlying hedonic price function is assumed to be linear. Transformation of the f_j' in equation (41) to derive the APAM adjustment

factor is analogous to the transformation of f_j to derive equation (34);
i.e.:

$$(42) \qquad g'_{ji} = \frac{f'_j X_{is}}{SP_i}$$

The adjustment factor for the Multiplicative Percentage Adjustments Method (MPAM), if the underlying hedonic price function is Cobb-Douglas, is as follows:

$$(43) \qquad h'_j = \frac{\ln K_{max} - \ln K_{min}}{\ln X_{jmax} - \ln X_{jmin}} \; .$$

where K_{max} = cost of the whole building with the maximum magnitude of the jth attribute found among the comparables and subject, and

K_{min} = cost of the whole building with the minimum magnitude of the jth attribute found among the comparables and subject.

Alternatively, if the underlying function is exponential, the adjustment factor for the MPAM is as follows:

$$(44) \qquad h'_j = \frac{\ln K_{max} - \ln K_{min}}{X_{jmax} - X_{jmin}} \; .$$

WEIGHTING SCHEMES

The adjustment grid methods result in a number of indicated values, one for the comparison of the subject with each comparable. These indicated values must be reconciled. Most appraisal texts indicate that the reconciliaton is not just a simple average in the sense of each weight being $1/n$.[15] Some recommend that the appraiser choose weights based on experience.[16] This recommendation is essentially devoid of content. Ratcliff suggests that the weights bear some relationship to how comparable each comparable is.[17] This is helpful, but still not quantitatively operational.

For the Additive Dollar Adjustments Method, reconciliation may be accomplished via a weighted average as follows:

$$(45) \qquad V_s = \sum_{i=1}^{n} w_i \widehat{SP}_s^i$$

where V_s = the reconciled value estimate,

w_i = the proportionate weight of the ith comparable, such that $w_i \geq 0$,

$\sum_{i=1}^{n} w_i = 1$, and

n = the number of comparables.

That is, the weights are non-negative and they must sum to unity. Unfortunately, mystery shrouds the derivation of these weights.

In order to develop a system in which the weights can be objectively determined, it is necessary to outline what it is that the weights ought to do. The weights ought to emphasize the indicated value from a comparable, relative to how comparable it is. Less comparability means a smaller weight and more comparability means a larger weight. The weighting scheme ought not to use the total (net) adjustment as an index of comparability because large plus and minus adjustments, indicating a lack of comparability, can net out to a small total adjustment. It is necessary to make use of the adjustments on each attribute in order to develop a sensible weighting scheme.

Four possible weighting schemes are presented. These four are schemes based on: (1) the absolute value of adjustments; (2) squared adjustments; (3) the use of distances between a subject and each comparable as weights; and (4) the use of a factor which avoids any comparable receiving zero weight.

There is no theoretically optimal weighting scheme. Whether one performs better than another is entirely an empirical question. Thus, the choice of a weighting scheme may be a matter of judgment tempered by experience.[18] However, with the use of any of these four schemes the weighting or reconciliation itself is an objective matter.

Absolute Value Weighting

Basing weights on the absolute value of each adjustment produces a weight as follows for the Additive Dollar Adjustments Method (ADAM):

$$(46) \qquad w_i = \frac{\sum_{k=1}^{n} \sum_{j=1}^{m} \left| f_j(X_{js}-X_{jk}) \right| - \sum_{j=1}^{m} \left| f_j(X_{js}-X_{ji}) \right|}{(n-1) \sum_{k=1}^{n} \sum_{j=1}^{m} \left| f_j(X_{js}-X_{jk}) \right|}$$

where $\sum_{k=1}^{n} \sum_{j=1}^{m} \left| f_j(X_{js}-X_{jk}) \right|$ = the sum of the absolute values of all adjustments made within a grid,

$\sum_{j=1}^{m} \left| f_j(X_{js}-X_{ji}) \right|$ = the sum of the absolute values of all adjustments made for comparable i,

m = the number of attributes for which adjustments are made, and

n = the number of comparables.

Weights derived in this way reflect the number and size of adjustments and ignore the reliability of adjustments.[19] For all three grid methods the range of the weights derived using absolute values of adjustments is from zero, if a particular comparable is the only one requiring adjustments, to $1/(n-1)$ if a particular comparable requires no adjustments. With the absolute value weighting scheme, the adjustment factors play no role in the prediction when the subject property's attributes fall between those of the comparables. Within this range, for the Additive

Dollar Adjustments Method the predicted selling price of the subject is equal to the weighted average of the selling prices of the comparables. However, the adjustment factors play an important role when the subject's attributes fall outside the range of the comparables. As the subject's attributes deviate from those of the comparables, the predicted selling price approaches the simple average of the indicated values from each of the comparables.[20] This is illustrated in Exhibit 1 for the situation in which $f_1 > (SP_2 - SP_1)/(X_{12} - X_{11})$. If the opposite obtains, the predicted selling price approaches the average of indicated values from the opposite directions shown in Exhibit 1.

Another potential problem with this scheme is that two comparables would be weighted equally if one had a lot of little adjustments which in absolute value equaled the few big adjustments for the other comparable. This problem, if it is one, can be solved by a different type of weighting scheme which treats the few large adjustments as indicating less comparability than the many small adjustments. This scheme is based on squared adjustments rather than the absolute value of adjustments.

Squared Weighting

Under the squared adjustments scheme, a weight for the Additive Dollar Adjustments Method is derived as follows:[21]

$$(47) \quad w_i^* = \frac{\sum\limits_{k=1}^{n} \sum\limits_{j=1}^{m} (f_j(X_{js} - X_{jk}))^2 - \sum\limits_{j=1}^{m} (f_j(X_{js} - X_{ji}))^2}{(n-1) \sum\limits_{k=1}^{n} \sum\limits_{j=1}^{m} (f_j(X_{js} - X_{jk}))^2} .$$
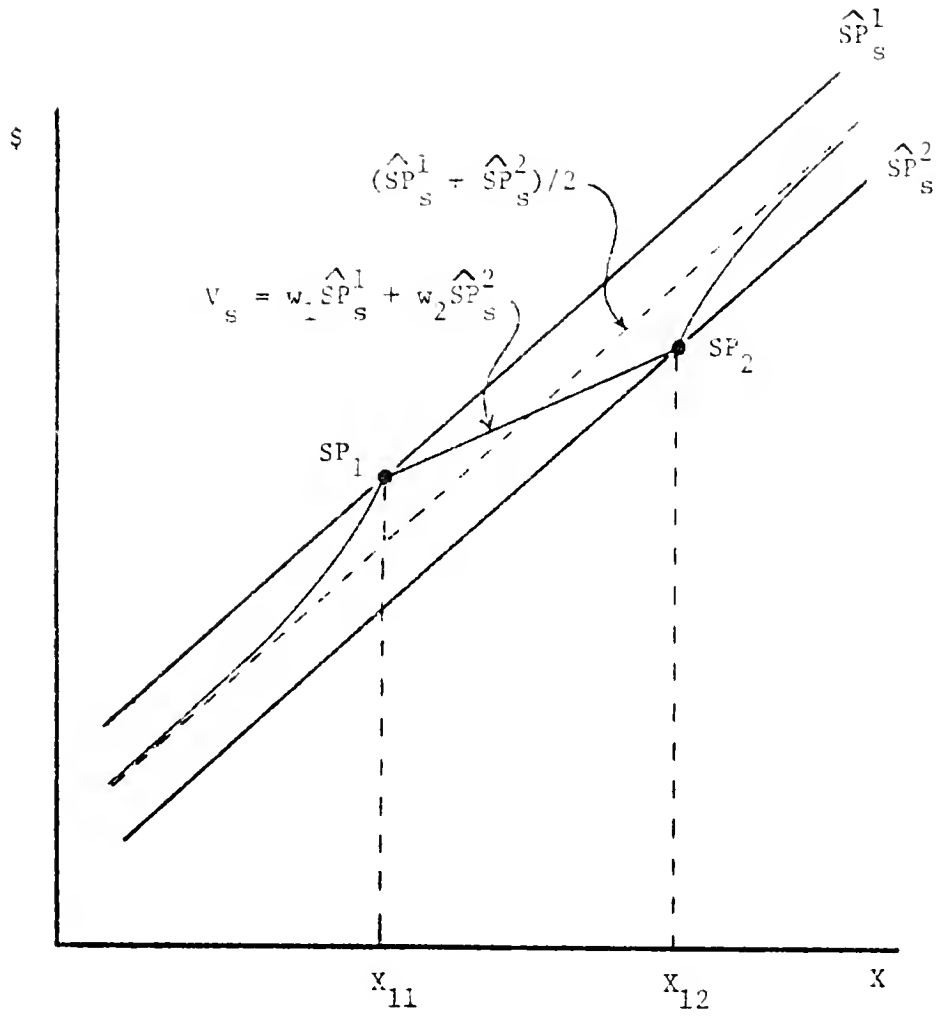
Exhibit 1

ABSOLUTE VALUE WEIGHTING

Again, the weights vary from 0 to $1/(n-1)$. And the weights are used in the same way.

In contrast to the weighting scheme utilizing absolute values, the adjustment factors matter in this weighting scheme even when the subject's attributes fall within the range of the comparables. Only a subject property having attributes exactly half way between the corresponding attributes of the comparables will have a reconciled value equal to the simple average of the selling prices of the comparables. Like the previous weighting scheme, the reconciled value approaches the simple average of the indicated values as the subject's attribute deviates from those of the comparables, in the limit. All this is illustrated in Exhibit 2. Note that Exhibit 2 illustrates a condition in which $f_1 > (SP_2 - SP_1)/(X_{12} - X_{11})$. If the opposite obtains, the predicted selling price approaches the average of indicated values from the opposite directions shown in Exhibit 2.

There are three inflection points on the function of the predicted selling price. One is always located at the mid-point corresponding to $\frac{X_{11} + X_{12}}{2}$. The locations of the other two inflection points depend on the values of X_{11} and X_{12}, but are always outside the interval from X_{11} to X_{12}. The values at these two inflection points are $X_{11} + \frac{1+\sqrt{3}}{2}(X_{12} - X_{11})$ and $X_{11} - \frac{\sqrt{3}-1}{2}(X_{12} - X_{11})$. The former is always greater than X_{12}, while the latter is less than X_{11}.

At the mid-point, the slope of the predicted selling price function will be affected by the values of X_{11}, X_{12} and f_1. The slope becomes positive only if the slope of the line connecting the two comparable SP and X combinations is at least twice the magnitude of the adjustment factor.

More generally, $\frac{dV_s}{dX_{1s}} \gtreqless 0$ if $\frac{2(SP_2 - SP_1)}{X_{12} - X_{11}} \gtreqless f_1$.

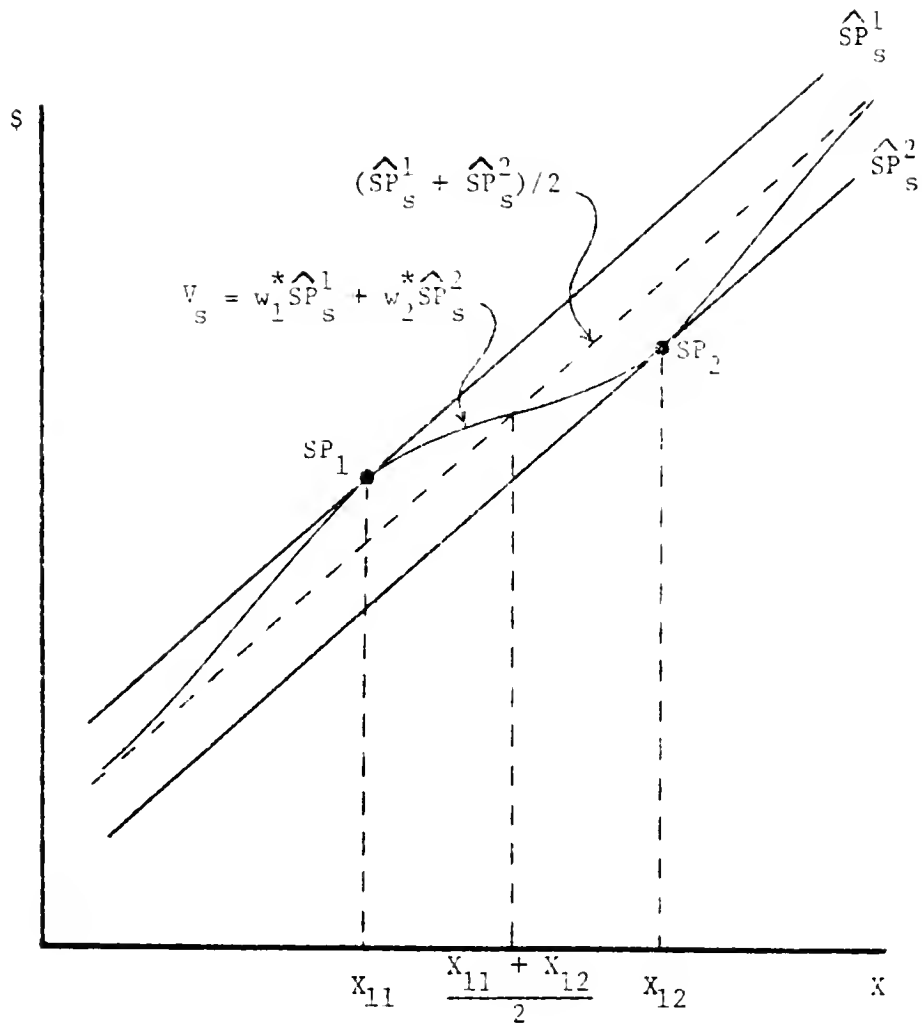Exhibit 2

SQUARED WEIGHTING

Distances as Weights

If the key variables not accounted for in making adjustments to comparables are related to locational differences, a weighting scheme based on distances between a subject and each comparable may be desirable. A weight for such a scheme could be derived as follows:

$$(43) \qquad w_i^{**} = \frac{\sum\limits_{k=1}^{n} d_k^2 - d_i^2}{(n-1) \sum\limits_{k=1}^{n} d_k^2}$$

where: d_i = distance between subject and ith comparable.

Of course, distances would not have to be squared but this is conventional practice for many similar applications.[22]

No Zero Weights

It could be argued that if a comparable is selected for use in an adjustment grid method that it should receive at least some weight in the reconciliation of indicated values. This could be accomplished using slightly modified versions of the equations developed for the schemes using absolute value and squared weighting. The only modification needed is to add a term, say Q, in the numerator and add a term, nQ, in the denominator. The term Q could represent any positive real number that would result in the desired minimum weight for a comparable that would otherwise have zero weight.[23]

GRID VS. REGRESSION

Under certain conditions it can be shown that grid-based predictions are conceptually superior to pure regression predictions. Pure regression prediction suffers from an omitted variable problem, the

solution to which may be the grid approach with weighted reconciliation of indicated values. The following analysis applies to the Additive Dollar Adjustments Method, but could be extended to the other two methods.

The true model is expressed as follows:

(49) $SP_i = \beta x_i + \rho z_i + \nu_i$,

where: β and ρ are vectors of coefficients,

x_i and z_i are vectors of property attributes, and

ν_i is the error term associated with the ith property.

The usual assumptions are

$$E(\nu_i | x_i, z_i) = 0$$
$$E(\nu_i \nu_j) = 0 \quad \text{if } i \neq j$$
$$= \sigma_\nu^2 \quad \text{if } i = j$$
$$E(SP_i | x_i, z_i) = \beta x_i + \rho z_i$$

The z_i's are difficult to measure and vary greatly across space. Many of them are neighborhood characteristics. Neighborhoods are small so sufficient observations usually cannot be found to use neighborhood dummy variables as proxies. Therefore, the estimation model is as follows:

(50) $SP_i = \hat{\beta} x_i + \hat{\mu}_i$,

where: $\hat{\mu}_i = \hat{\rho} z_i + \hat{\nu}_i$.

A prediction based solely on regression is as follows:

(51) $\widehat{SP}_s = \hat{\beta} x_s$.

An indicated value from the Additive Dollar Adjustments Method is as follows:

$$(52) \qquad \widehat{SP}_s^1 = SP_1 + \hat{\beta}(x_s - x_1).$$

Expanding and substituting from equation (50) for $i = 1$ yields:

$$(53) \qquad \widehat{SP}_s^1 = \hat{\beta}x_s + \hat{\mu}_1.$$

The reconciled estimate of market value is as follows:

$$(54) \qquad V_s = w_1\widehat{SP}_s^1 + w_2\widehat{SP}_s^2.$$

Using the definition of $\hat{\mu}_i$ and equation (53), the reconciled value is as follows:

$$(55) \qquad V_s = \hat{\beta}x_s + w_1(\hat{\rho}z_1 + \hat{\nu}_1) + w_2(\hat{\rho}z_2 + \hat{\nu}_2).$$

This method produces an unbiased prediction of the selling price of the subject property if

$$(56) \qquad w_1z_1 + w_2z_2 = z_s.$$

The weighted average of the excluded variables for the comparables must equal the corresponding variable for the subject. Assuming that this is the case, taking the expected values of equations (51) and (55), and comparing these expectations to the true model makes clear the superiority of the grid approach over a pure regression approach under the conditions hypothesized.

The expected value of the reconciled value from grid-based predictions is as follows:

(57) $\quad E(V_s) = \hat{\beta}x_s + \hat{\rho}z_s.$

The expected value of the predicted selling price based solely on regression is as follows:

(58) $\quad E(SP_s) = \hat{\beta}x_s.$

Comparing equations (57) and (58) to the true model represented by equation (49), it can be seen that only the grid-based prediction is unbiased given the restriction of equation (56).

The presence and direction of bias in the grid method may be viewed in a less restrictive context. One may assume a relationship between the comparable and subject z's that corresponds to a relationship between the comparable and subject x's. For example,

$$z_s = a_1z_1 \text{ while } x_s = a_1x_1, \text{ and}$$
$$z_s = a_2z_2 \text{ while } x_s = a_2x_2.$$

If this is the case, and the absolute value weighting is applied, the bias can be shown to depend on the relationships of the comparable to the subject z's.

First, if $z_1 \leq z_s \leq z_2$, equation (56) always holds. That is, the grid method is unbiased. Second, if $z_1 < z_2 < z_s$, then

(59) $\quad w_1z_1 + w_2z_2 = z_s\left[\dfrac{(a_1-1) + (a_2-1)}{a_2(a_1-1) + a_1(a_2-1)}\right] = wz_s.$

Since $a_1 > a_2 > 1$, $0 < w < 1$, the grid method has downward bias. Nevertheless, it is less biased than the pure regression method since some of the effects of neighborhood characteristics (i.e., the z's) are captured

by the grid method. Finally, if $z_s < z_1 < z_2$, $w_1 z_1 + w_2 z_2 < z_s$, the grid method involves upward bias in estimating z_s. As shown in equation (59), $w > 1$, since $1 > a_1 > a_2 > 0$. This method will be less biased than the pure regression method, if a_1 and a_2 are not extremely small.

Alternatively, if the squared weighting is applied, then equation (59) becomes equation (60):

$$(60) \qquad w_1 z_1 + w_2 z_2 = z_s \; \frac{a_2(1 - a_1)^2 + a_1(1 - a_2)^2}{a_2^2(1 - a_1)^2 + a_1^2(1 - a_2)^2} \; = w^* z_s.$$

First, if $z_1 < z_2 < z_s$ or $z_1 < z_s < \frac{1}{2}(z_1 + z_2)$, then $0 < w^* < 1$. Second, if $z_2 > z_s > \frac{1}{2}(z_1 + z_2)$, then $1 < w^* < 2$. Third, if $z_s < z_1 < z_2$, then $w^* > 1$. Again, if a_1 and a_2 are not extremely small, the grid method will be less biased than the pure regression method. Finally if $z_s = z_1$, $z_s = z_2$ or $z_s = \frac{1}{2}(z_1 + z_2)$, then again equation (56) always holds.

CONCLUSIONS

Hedonic price functions underly both grid and regression approaches to appraisal. Based on comparison to a hypothetical true model, grid-based prediction is shown to be unbiased under very restrictive conditions. On the other hand, pure regression prediction is shown to suffer bias that originates from an omitted variable problem. More generally, the grid method is shown to be less biased than the pure regression method.

The rational selection of one adjustment grid method over another should be based on the nature of the functional relationship between selling price and property attributes. Estimation of adjustment factors

also should be consistent with this relationship. The use of matched pairs to estimate adjustment factors is unreliable. Regression provides a more reliable method if sufficient data is available. If the housing market is in equilibrium, cost data may be used to estimate adjustment factors.

The choice of a weighting scheme for reconciliation of indicated values generated by a grid method may be a matter of subjective judgment based on experience since there is no theoretically optimal scheme. However, the weighting or reconciliation itself can be an objective matter if one of the weighting schemes presented here is used.

NOTES

[1]For a detailed development of the theory of hedonic price functions, see: Rosen [11].

[2]The term Additive is used here instead of Plus and Minus (or) Plus or Minus which appear in some references. Also, the term Multiplicative is used here instead of Cumulative which appears in some references. The terms Additive and Multiplicative are believed to be more descriptive of the procedures in each method.

[3]See, for example: Green [4], p. 178 and Unger [14], p. 23.

[4]See: Friedman and Ordway [3], p. 310.

[5]See: Kinnard and Boyce [5], p. 10-20. (Also, see: Shenkel [12], p. 150.)

[6]See: Miller and Gilbeau [8], p. 123.

[7]See: Bloom and Harrison [2], p. 262.

[8]See: Green [4], p. 178.

[9]See: Ring [10], p. 121.

[10]See: Bloom and Harrison [2], pp. 264 and 265.

[11]The magnitudes of the attributes for the subject property show up in the adjustment factors because they are used as the base for the percentage differences in equation (13). This is consistent with Kinnard and Boyce [5], p. 10-20.

[12]Note that $\ln \left(\frac{X_{1s}}{X_{11}}\right)$ equals $\ln X_{1s} - \ln X_{11}$.

[13]See, for example: American Institute of Real Estate Appraisers (AIREA) Textbook Revision Subcommittee [1], p. 286; Kinnard and Boyce [5], p. 10-21; and, Bloom and Harrison [2], p. 248.

[14]See, for example: AIREA Textbook Revision Subcommittee [1], p. 59 and Chapter 12; and, Kinnard and Boyce [5], pp. 13-12 to 13-16.

[15]An exception that advocates use of a simple average is: Smith [13], p. 42.

[16]See, for example: Ratcliff [9], pp. 156-161; and Ring [10], pp. 136-138.

[17]See: Ratcliff [9], p. 160.

[18]The type of experience implied is that experience related to empirical testing of various weighting schemes in a particular market.

[19]One way to incorporate reliability is to multiply each adjustment factor by the ratio of the selected t-ratio for a particular level of confidence, say 95%, to the computed t-ratio corresponding to the factor. For example, absolute value weighting with the Additive Dollar Adjustment Method would be as follows:

$$w_i = \frac{\sum\limits_{k=1}^{n} \sum\limits_{j=1}^{m} \left| \frac{f_j t^*}{t_j}(X_{js} - X_{jk}) \right| - \sum\limits_{j=1}^{m} \left| \frac{f_j t^*}{t_j}(X_{js} - X_{ji}) \right|}{(n-1) \sum\limits_{k=1}^{n} \sum\limits_{j=1}^{m} \left| \frac{f_j t^*}{t_j}(X_{js} - X_{jk}) \right|}$$

[20]The preceding statements are true for the Multiplicative Percentage Adjustments Method if the natural logarithm of selling price or indicated value is substituted for selling price or indicated value.

[21]Statements in this section are correct for the Additive Dollar Method. Again, to be true for the Multiplicative Percentage Adjustments Method the natural logarithm of selling price or indicated value must be substituted.

[22]See, for example: Messner, et. al. [7], pp. 70 and 71.

[23]The appropriate Q is a function of the minimum weight (w_{min}) that is desired. For example, under the squared adjustments scheme a Q for the Additive Dollar Adjustments Method is derived as follows:

$$Q = \frac{(n-1)}{(1 - nw_{min})} \sum\limits_{k=1}^{n} \sum\limits_{j=1}^{m} (f_j(X_{js} - X_{jk}))^2 (w_{min}).$$

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